Carleton UNIVERSITY
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Mathematics Tutorial Series
Differential Calculus #16

Examples III

[1] Calculate the derivative of each.

a) $y = \tan(3 + \log x)$
b) $y = (\cos x)^x$
c) $y = x^{2\sin x}$
d) $y = \sin^{-1}(\sqrt{1-x})$
e) $y = \left(1 - \sqrt{x}\right)^{x+1}$
f) $y = x\sin^{-1} x + \sqrt{1-x^2}$ for $-1 < x < 1$.
g) $y = x\tan^{-1} x - \log(1 + x^2)$

[2] Consider the curve with the equation
$x^2 + y^2 - 9 = 4xy^2$.
Calculate the slope of the tangent line at the point $(x, y) = (0,3)$.

[3] Consider the curve defined by $x^2 - xy + y^2 = 9$.
Determine a formula for the slope of the tangent line at a general point.
Calculate the slopes of the tangents at the two points where the curve crosses the x-axis. Show that these lines are parallel.

[4] Let $f(x) = (x + 5)e^{-2x}$.
a) Describe the intervals where f is positive and where it is negative.
b) Calculate the derivative.
c) Find the intervals where f is increasing and decreasing.

[5] Same again for $f(x) = \frac{x^2}{x^2-4}$.

DON"T LOOK AT THE ANSWERS
UNTIL YOU HAVE TRIED TO SOLVE THE PROBLEMS.

Solutions

[1] Calculate the derivative of each.

a) $y = \tan(3 + \log x)$

This is $3 + \log x$ substituted into $\tan x$ so we use the Chain Rule.
$y' = \sec^2(3 + \log x)\,(3 + \log x)' = \sec^2(3 + \log x)\left(\frac{1}{x}\right)$

b) $y = (\cos x)^x$

Take logs on both sides then differentiate implicitly.

$$\log y = x \log \cos x$$

Take the derivative:

$$\frac{1}{y}y' = x' \log \cos x + x(\log \cos x)'$$

$$= \log \cos x + x\frac{1}{\cos x}(\cos x)'$$

$$= \log \cos x - x\frac{\sin x}{\cos x}$$

$$= \log \cos x - x \tan x$$

So

$$y' = (\log \cos x - x \tan x)(\cos x)^x$$

c) $y = x^{2 \sin x}$

Again, logarithmic differentiation works well.

$$\log y = 2 \sin x \log x$$

Take the derivative:

$$\frac{1}{y}y' = 2\left(\cos x \log x + \sin x\left(\frac{1}{x}\right)\right)$$

So

$$y' = 2\left(\cos x \log x + \sin x\left(\frac{1}{x}\right)\right)x^{2 \sin x}$$

d) $\boldsymbol{y = \sin^{-1}(\sqrt{1-x})}$

$$y' = \frac{1}{\sqrt{1-\left(\sqrt{1-x}\right)^2}}(\sqrt{1-x})'$$

$$= \frac{1}{\sqrt{1-\left(\sqrt{1-x}\right)^2}}\frac{1}{2\sqrt{1-x}}(-1)$$

$$= \frac{1}{2}\frac{1}{\sqrt{x}}\frac{-1}{\sqrt{1-x}}$$

e) $\boldsymbol{y = \left(1-\sqrt{x}\right)^{x+1}}$

Logarithmic differentiation.

$$\log y = (x+1)\log(1-\sqrt{x})$$

Differentiate.

$$\frac{1}{y}y' = (x+1)'\log(1-\sqrt{x}) + (x+1)(\log(1-\sqrt{x}))'$$

$$= \log(1-\sqrt{x}) + (x+1)\frac{1}{1-\sqrt{x}}\frac{-1}{2\sqrt{x}}$$

So

$$y' = (\log(1-\sqrt{x}) + (x+1)\frac{1}{1-\sqrt{x}}\frac{-1}{2\sqrt{x}})\left(1-\sqrt{x}\right)^{x+1}$$

f) $\boldsymbol{y = x\sin^{-1}x + \sqrt{1-x^2}}$

The condition that $-1 < x < 1$ is needed so that $\sin^{-1}x$ and $\sqrt{1-x^2}$ can be calculated. Conditions like this could have been added to the other questions. Also an exam question might ask you: specify the domain of the function.

We use product rule for the first term then chain rule for the second.

$$y' = \sin^{-1}x + x\,\frac{1}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{1-x^2}}(-2x)$$

$$= \sin^{-1}x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}$$

$$= \sin^{-1}x$$

g) $\boldsymbol{y = x\tan^{-1}x - \log(1+x^2)}$

We use product rule for the first term then chain rule for the second.

$$\begin{aligned} y' &= \tan^{-1}x + x\frac{1}{1+x^2} - \frac{1}{1+x^2}(2x) \\ &= \tan^{-1}x - \frac{x}{1+x^2} \end{aligned}$$

[2] Consider the curve with the equation $\boldsymbol{x^2 + y^2 - 9 = 4xy^2}$**.**
Calculate the slope of the tangent line at the point $\boldsymbol{(x,y) = (0,3)}$**.**

Solution:

The slope of the tangent line is the value of the derivative $\frac{dy}{dx}$.
Use implicit differentiation.

$$2x + 2y\frac{dy}{dx} = 4y^2 + 4x(2y)\frac{dy}{dx}$$

Solve for $\frac{dy}{dx}$:

$$(2y - 8xy)\frac{dy}{dx} = 4y^2 - 2x$$

$$\frac{dy}{dx} = \frac{4y^2 - 2x}{2y - 8xy}$$

Then the slope of the tangent line at $(x,y) = (0,3)$ is $\frac{36-0}{6-0} = 6$.

[3] Consider the curve defined by $x^2 - xy + y^2 = 9$.
Determine a formula for the slope of the tangent line at a general point. Calculate the slopes of the tangents at the two points where the curve crosses the x-axis. Show that these lines are parallel.

Solution

Use implicit differentiation.

$$2x - y - xy' + 2yy' = 0$$

Hence

$$y' = \frac{y - 2x}{2y - x}$$

The curve crosses the x-axis when $y = 0$ hence when $x^2 = 9$.
Thus the intersection points are $(x, y) = (3,0)$ and $(x, y) = (-3,0)$.

The slope of the tangent lines at either point is $+2$. Lines with the same slope are parallel so we have ansered the question.

[4] Let $f(x) = (x + 5)e^{-2x}$.

a) **Describe the intervals where f is positive and where it is negative.**
b) **Calculate the derivative.**
c) **Find the intervals where f is increasing and decreasing.**

Solution

a) The values of the exponential function are always positive. So f is positive when $x + 5$ is positive and negative when $x + 5$ is negative.

So $f(x) > 0$ when $x > -5$ and $f(x) < 0$ when $x < -5$.

b) Use the product rule then chain rule.

$$f' = e^{-2x} + (x + 5)(-2)e^{-2x} = (-2x - 9)e^{-2x}$$

c) The function is increasing when $f' > 0$ and decreasing when $f' < 0$.

Again, the exponential function has only positive values, so we look at $(-2x - 9)$.

We see that $f' > 0$ and f is increasing when $-2x - 9 > 0$ so when $x < -9/2$.

And $f' < 0$ and f is decreasing when $-2x - 9 < 0$ so when $x > -9/2$.

[5] Same again for $f(x) = \frac{x^2}{x^2-4}$.

a) Since x^2 is never negative, the function $f(x) < 0$ only when $x^2 - 4 < 0$.

Hence $f(x)$is positive except when $-2 < x < +2$.
Note that this function has vertical asymptotes (poles) at ± 2.

b) Use the quotient rule.

$$f'(x) = \frac{(2x)(x^2-4) - x^2(2x)}{(x^2-4)^2} = \frac{-8x}{(x^2-4)^2}$$

c) The function is increasing when $f' > 0$ and decreasing when $f' < 0$.

In this case the denominator of the derivative is always positive or zero. Thus $f' > 0$
So f is increasing when $-8x > 0$. This happens when $x < 0$.
Also f is decreasing when $-8x < 0$. This happens when $x > 0$.